Lecture 28

In this lecture we begin our study of probability.

Probability

Suppose that an experiment or activity involves certain <u>events</u> or <u>outcomes</u> which may possibly occur. The basic or <u>elementary</u> outcomes are also called the "atomic events", and the set of all such outcomes is called the <u>sample space.</u> These outcomes are equally likely.

<u>E.g.</u> Suppose a die is thrown. The sample space is

$$\{1, 2, 3, 4, 5, 6\}.$$

This set displays the six elementary outcomes.

But we could also speak of other outcomes such as "a score of 3 or more".

Such outcomes are called <u>compound</u> outcomes or events, because they can be described in terms of the elementary outcomes.

Often we classify some outcomes as "desirable" for the purposes of a particular experiment. Then the set of desirable outcomes D is a subset of the sample space S.

For example,

$$D = \{3, 4, 5, 6\}$$

is the set of desirable outcomes if we want a score of "3 or more" when tossing a die.

Then the probability of getting a desired outcomes is as follows:

$$Pr(D) = \frac{|D|}{|S|}$$

In our current example we get

$$Pr(3 \text{ or more}) = \frac{4}{6}$$

$$= \frac{2}{3}.$$

Since $\emptyset \subseteq D \subseteq S$ it is clear that

$$|\emptyset| \leq |D| \leq |S|$$

whence $$\frac{|\emptyset|}{|S|} \leq \frac{|D|}{|S|} \leq \frac{|S|}{|S|} \; ;$$

i.e., $$0 \leq Pr(D) \leq 1 .$$

An impossible outcome is one whose probability is 0, and a certain outcome (in the sense that it certainly must occur) is an event whose probability is equal to 1.

If A and B are two outcomes, then we can regard A and B as sets of elementary outcomes and therefore as subsets of the sample space S.

We know that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

If also A and B are disjoint then we have

$$|A \cup B| = |A| + |B|$$

so that

$$\begin{aligned}
\Pr(A \cup B) &= \frac{|A \cup B|}{|S|} \\
&= \frac{|A| + |B|}{|S|} \\
&= \frac{|A|}{|S|} + \frac{|B|}{|S|} \\
&= \Pr(A) + \Pr(B).
\end{aligned}$$

This result can be extended to any finite union of pairwise disjoint sets.

This means that given several disjoint events, the probability of A_1 or A_2 or ... or A_k is equal to the sum of their individual probabilities.

For example, if we cast a die then we could describe some possible outcomes as follows:

A_1 means "a score of 2 or less";

A_2 means "a score of 4"; and

A_3 means "a score of 5 or more".

These three events are pairwise disjoint and have individual probabilities

$$Pr(A_1) = Pr(1 \text{ or } 2) = \frac{2}{6} = \frac{1}{3},$$

$$Pr(A_2) = \frac{1}{6} \quad \text{and}$$

$$Pr(A_3) = Pr(5 \text{ or } 6) = \frac{2}{6} = \frac{1}{3}.$$

So the probability of A_1 or A_2 or A_3 is

$$Pr(A_1 \cup A_2 \cup A_3) = Pr(A_1) + Pr(A_2) + Pr(A_3)$$

$$= \frac{2}{6} + \frac{1}{6} + \frac{2}{6}$$

$$= \frac{5}{6}.$$

If two outcomes A and B are not necessarily disjoint, then the probability of A or B is

$$Pr(A \cup B) = \frac{|A \cup B|}{|S|}$$

$$= \frac{|A| + |B| - |A \cap B|}{|S|}$$

$$= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|}$$

$$= Pr(A) + Pr(B) - Pr(A \cap B).$$

For example, let A be the outcome of "an even number" and let B be the outcome of "a number greater than 3" when a die is cast.

Then $A = \{2, 4, 6\}$ and $B = \{4, 5, 6\}$ so that

$$A \cap B = \{4, 6\}.$$

So

$$\begin{aligned} Pr(A \cup B) &= Pr(A) + Pr(B) - Pr(A \cap B) \\ &= \frac{3}{6} + \frac{3}{6} - \frac{2}{6} \\ &= \frac{4}{6} = \frac{2}{3}. \end{aligned}$$

It's convenient to discuss what happens when a desired event <u>doesn't</u> occur. If E is an event then we let $\sim E$ be the event "not E". So

$$\sim E = S \setminus E \quad [= S - E \text{ in some texts}].$$

This means that $\sim E$ is the complement of E relative to S.

Then

$$Pr(\sim E) = 1 - Pr(E).$$

For example, if E is the event of scoring "5 or more" when throwing a die then $\sim E$ is the event of scoring "less than 5".

Then

$$\begin{aligned} Pr(\sim E) &= 1 - Pr(E) \\ &= 1 - \frac{2}{6} \\ &= \frac{4}{6}. \end{aligned}$$

We can confirm this by observing that $E = \{5, 6\}$ so that $\sim E = \{1, 2, 3, 4\}$.

Having established the basic theory, it's time to look at more complicated examples.

E.g. Suppose a die is thrown twice. What is the sample space of all possible outcomes?

$\underline{\text{Sol}^{\underline{n}}}$

$$S = \{(1,1), (1,2), \ldots, (1,6), \\ (2,1), (2,2), \ldots, (2,6), \\ \vdots \\ (6,1), (6,2), \ldots, (6,6)\}$$

This is, in fact, the Cartesian product of the set $\{1, 2, 3, 4, 5, 6\}$ with itself.

E.g. (cont.)

What's the probability that the sum of the two scores is an even number?

$\underline{\text{Sol}^{\underline{n}}}$

Let the desired outcome be A. It can happen in 18 ways (when both scores are even or both are odd). So the probability is

$$\Pr(A) = \frac{18}{36} = \frac{1}{2}.$$

E.g. (cont.)

What's the probability the two scores are relatively prime?

Sol$^{\underline{n}}$

This is harder.

With 1 as first coordinate there are 6 ordered pairs.

"	2	"	"	"	"	3	" " .
"	3	"	"	"	"	4	" " .
"	4	"	"	"	"	3	" " .
"	5	"	"	"	"	5	" " .
"	6	"	"	"	"	2	" " .

So the total no. of possibilities is 23.

$\therefore$ the prob. is $\dfrac{23}{36}$.

E.g. (cont.)

What's the probability the two scores are not relatively prime?

Sol$^{\underline{n}}$

$$1 - \frac{23}{36} = \frac{13}{36}$$

E.g. (cont.)

What's the probability the two scores add up to an even number or are relatively prime?

Sol$^{\underline{n}}$

Let A mean the sum is even.

Let B mean the two scores are relatively prime.

Then

$$A \cap B = \{(1,1), (1,3), (1,5), (3,1), (3,5), (5,1), (5,3)\}.$$

$$\therefore \Pr(A \cup B) = \frac{18}{36} + \frac{23}{36} - \frac{7}{36}$$

$$= \frac{34}{36}$$

<u>Note</u>

$$\Pr[\sim(A \cup B)] = \frac{2}{36}$$

This suggests that there are only two elementary outcomes which satisfy neither A nor B.

They are $(3,6)$ and $(6,3)$.